

Exploration 2-4a

$$f(x) = \begin{cases} x+1 & x < 2 \\ (x-5)^2 & x \geq 2 \end{cases}$$

$$g(x) = \begin{cases} x+1 & x \leq 2 \\ \frac{1}{3}(x-5)^2 & x \geq 2 \end{cases}$$



$$f(x) = \begin{cases} x+1 & x < 2 \\ k(x-5)^2 & x \geq 2 \end{cases}$$

$$\lim_{x \rightarrow 2^-} f(x) = 2+1 = 3$$

$$\lim_{x \rightarrow 2^+} f(x) = k(3-5)^2 = 9k$$

$$3 = 9k$$

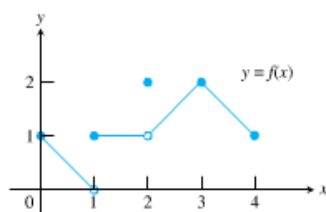
$$k = \frac{1}{3}$$

Continuity at a Point

Function f is continuous at $x = c$ if and only if

1. $f(c)$ exists,
2. $\lim_{x \rightarrow c} f(x)$ exists, and (limit from the left must = limit from the right)
3. $\lim_{x \rightarrow c} f(x) = f(c)$.

Continuity at a Point



domain $[0, 4]$

EXAMPLE 1 Investigating Continuity

Find the points at which the function f in Figure 2.18 is continuous, and the points at which f is discontinuous.

Continuous : $x = 0, x = 3, x = 4$

discontinuous: $x = 1, x = 2$

continuous: $[0, 1) \cup (1, 2) \cup (2, 4]$

discontinuous at $x = 1$ & $x = 2$

DEFINITION Continuity at a Point

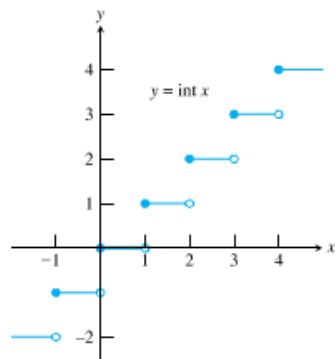
Interior Point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x \rightarrow a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \rightarrow b^-} f(x) = f(b), \quad \text{respectively.}$$

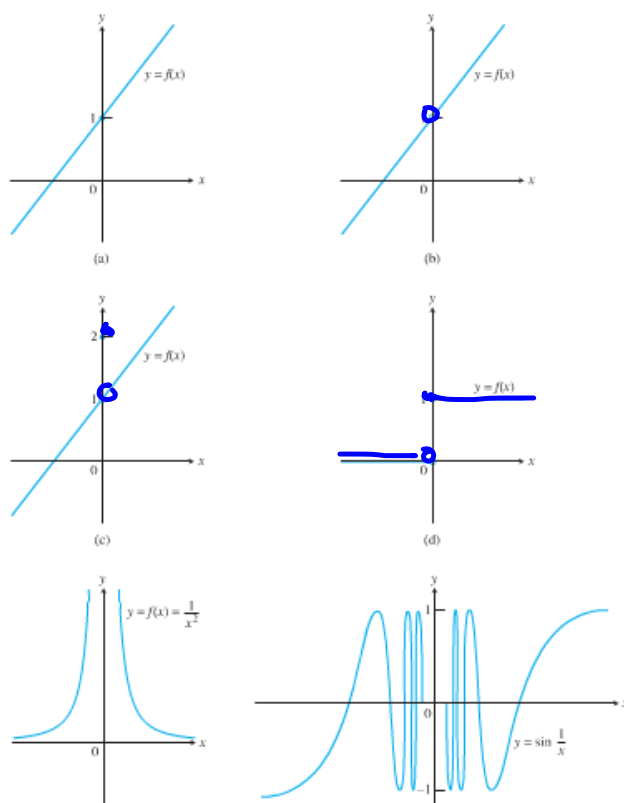
If a function f is not continuous at a point c , we say that f is **discontinuous** at c and c is a **point of discontinuity** of f . Note that c need not be in the domain of f .

**EXAMPLE 2 Finding Points of Continuity and Discontinuity**

Find the points of continuity and the points of discontinuity of the greatest integer function (Figure 2.20).

Figure 2.21 is a catalog of discontinuity types. The function in (a) is continuous at $x = 0$. The function in (b) would be continuous if it had $f(0) = 1$. The function in (c) would be continuous if $f(0)$ were 1 instead of 2. The discontinuities in (b) and (c) are **removable**. Each function has a limit as $x \rightarrow 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in (d)–(f) of Figure 2.21 are more serious: $\lim_{x \rightarrow 0} f(x)$ does not exist and there is no way to improve the situation by changing f at 0. The step function in (d) has a **jump discontinuity**: the one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in (e) has an **infinite discontinuity**. The function in (f) has an **oscillating discontinuity**: it oscillates and has no limit as $x \rightarrow 0$.



EXPLORATION 1 Removing a Discontinuity

$$\text{Let } f(x) = \frac{x^3 - 7x - 6}{x^2 - 9}.$$

1. Factor the denominator. What is the domain of f ?
2. Investigate the graph of f around $x = 3$ to see that f has a removable discontinuity at $x = 3$.
3. How should f be defined at $x = 3$ to remove the discontinuity? Use zoom-in and tables as necessary.
4. Show that $(x - 3)$ is a factor of the numerator of f , and remove all common factors. Now compute the limit as $x \rightarrow 3$ of the reduced form for f .
5. Show that the *extended function*

$$g(x) = \begin{cases} \frac{x^3 - 7x - 6}{x^2 - 9}, & x \neq 3 \\ 10/3, & x = 3 \end{cases}$$

is continuous at $x = 3$. The function g is the **continuous extension** of the original function f to include $x = 3$.

Now try Exercise 25.

Continuity on an Interval

- Function f is continuous on an interval of x -values if and only if it is continuous at each value of x in that interval. At the endpoints of a closed interval, only the one-sided limits need to equal the function value.

Continuous Functions

A function is **continuous on an interval** if and only if it is continuous at every point of the interval. A **continuous function** is one that is continuous at every point of its domain. A continuous function need not be continuous on every interval. For example, $y = 1/x$ is not continuous on $[-1, 1]$.

EXAMPLE 3 Identifying Continuous Functions

The reciprocal function $y = 1/x$ (Figure 2.22) is a continuous function because it is continuous at every point of its domain. However, it has a point of discontinuity at $x = 0$ because it is not defined there.

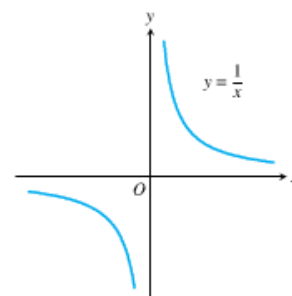


Figure 2.22 The function $y = 1/x$ is continuous at every value of x except $x = 0$. It has a point of discontinuity at $x = 0$. (Example 3)

THEOREM 6 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Products:* $f \cdot g$
4. *Constant multiples:* $k \cdot f$, for any number k
5. *Quotients:* f/g , provided $g(c) \neq 0$

Composites

All composites of continuous functions are continuous. This means composites like

$$y = \sin(x^2) \quad \text{and} \quad y = |\cos x|$$

are continuous at every point at which they are defined. The idea is that if $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is continuous at $x = c$ (Figure 2.23). In this case, the limit as $x \rightarrow c$ is $g(f(c))$.

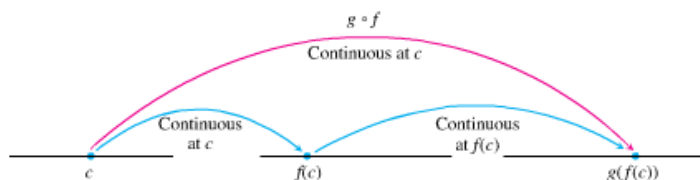


Figure 2.23 Composites of continuous functions are continuous.

THEOREM 7 Composite of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c .

EXAMPLE 4 Using Theorem 7

Show that $y = \left| \frac{x \sin x}{x^2 + 2} \right|$ is continuous.

$$f(x) = |x| \quad \text{continuous}$$

$$g(x) = \frac{x}{x^2 + 2} \quad \text{continuous}$$

$$h(x) = \sin x \quad \text{continuous}$$

$$y = f(g(x) \cdot h(x))$$

continuous

Exploration 2-6m

THEOREM 8 The Intermediate Value Theorem for Continuous Functions

A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

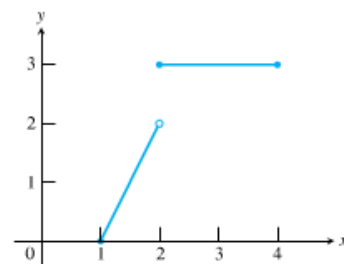
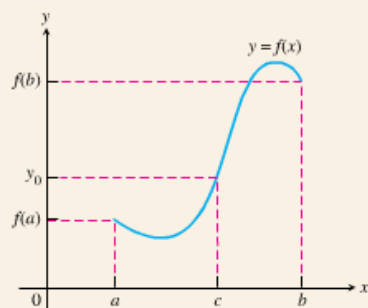


Figure 2.25 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \leq x < 2 \\ 3, & 2 \leq x \leq 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

Grapher Failure

In connected mode, a grapher may conceal a function's discontinuities by portraying the graph as a connected curve when it is not. To see what we mean, graph $y = \text{int}(x)$ in a $[-10, 10]$ by $[-10, 10]$ window in both connected and dot modes. A knowledge of where to expect discontinuities will help you recognize this form of grapher failure.

The continuity of f on the interval is essential to Theorem 8. If f is discontinuous at even one point of the interval, the theorem's conclusion may fail, as it does for the function graphed in Figure 2.25.

A Consequence for Graphing: Connectivity Theorem 8 is the reason why the graph of a function continuous on an interval cannot have any breaks. The graph will be **connected**, a single, unbroken curve, like the graph of $\sin x$. It will not have jumps like those in the graph of the greatest integer function $\text{int } x$, or separate branches like we see in the graph of $1/x$.

Most graphers can plot points (*dot mode*). Some can turn on pixels between plotted points to suggest an unbroken curve (*connected mode*). For functions, the connected format basically assumes that outputs vary *continuously* with inputs and do not jump from one value to another without taking on all values in between.